

SOLITARY WAVE SOLUTIONS AND THEIR VELOCITY SELECTIONS AND PROHIBITIONS FOR A GENERAL BOUSSINESQ TYPE FLUID MODEL

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ABSTRACT. The possible solitary wave solutions for a general Boussinesq (GBQ) type fluid model are studied analytically. After proving the non-Painlevé integrability of the model, the first type of exact explicit travelling solitary wave with a special velocity selection is found by the truncated Painlevé expansion. The general solitary waves with different travelling velocities can be studied by casting the problems to the Newtonian quasi-particles moving in some proper one dimensional potential fields. For some special velocity selections, the solitary waves possess different shapes, say, the left moving solitary waves may possess different shapes and/or amplitudes with those of the right moving solitons. For some other velocities, the solitary waves are completely prohibited. There are three types of GBQ systems (GBQSs) according to the different selections of the model parameters. For the first type of GBQS, both the faster moving and lower moving solitary waves allowed but the solitary waves with “middle” velocities are prohibit. For the second type of GBQS all the slower moving solitary waves are completely prohibit while for the third type of GBQS only the slower moving solitary waves are allowed.

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1. INTRODUCTION

The study of the Korteweg de-Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{1}$$

has been an interesting issue since the discovery of soliton. Its exact solution[1] can be used to describe the overtaking collision of soliton on a uniform layer of water, but the solution is only physically meaningful for the unidirectional soliton. All the left moving solitons with zero boundary conditions are prohibited.

In Ref.[2], three sets of model equations are derived for modelling nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters

of uniform depth. A good understanding of all solutions of these models are helpful for coastal and civil engineers to apply the nonlinear water wave model in a harbor and coastal design. Therefore, finding more types of solutions of these equations are fundamental interest in fluid dynamics. For the various different models, a point of central interest is to examine and compare the solitary wave solutions. For the case of plane motion, the three set of models can be rewritten as the Boussinesq class after omitting the higher order terms:

(A). The $\{\bar{u}, v\}$ system, – the depth-mean velocity basis

$$\begin{aligned}\bar{u}_t + \bar{u}\bar{u}_x + v_x &= \frac{1}{3}\bar{u}_{xxt} \\ v_t + [(1+v)\bar{u}]_x &= 0.\end{aligned}\tag{2}$$

(B). The $\{\underline{u}, v\}$ system, – the bottom variable basis

$$\begin{aligned}\underline{u}_t + \underline{u}\underline{u}_x + v_x &= \frac{1}{2}\underline{u}_{xxt} \\ v_t + [(1+v)\underline{u}]_x &= \frac{1}{6}\underline{u}_{xxx}.\end{aligned}\tag{3}$$

(C). The $\{\hat{u}, v\}$ system, – the surface variable basis

$$\begin{aligned}\hat{u}_t + \hat{u}\hat{u}_x + v_x &= 0 \\ v_t + [(1+v)\hat{u}]_x &= -\frac{1}{3}\hat{u}_{xxx}.\end{aligned}\tag{4}$$

In the Boussinesq types of systems (2)-(4), the field v is wave elevation and \bar{u} , \underline{u} and \hat{u} are depth-mean, bottom and surface velocities respectively.

To study the systems (2)-(4) uniformly, we can extended them to the following generalized Boussinesq system (GBQS):

$$\begin{aligned}u_t + uu_x + v_x &= c_1 u_{xxt} \\ v_t + [(1+v)u]_x &= c_2 u_{xxx}.\end{aligned}\tag{5}$$

The system (4) has been shown to be integrable and has Hamiltonian structures[3, 4]. The exact solitary wave and periodic wave solutions of (4) have also been obtained by many authors[5]. Especially, Zhang and Li[6] presented a theory of bidirectional solitons on water by using integrable Boussinesq surface-variable equation (4). However the integrability of the other two systems (2) and (3) or generally the GBQS (5) is not known. In this paper, we will show that the GBQS is non-Painlevé integrable generally except for a special equivalent case of (4). Though the model is non-Painlevé integrable, it is still possible and useful to give out some interesting solitary wave solutions. In this paper, we are concentrate on to study the exact solitary wave solutions of the GBQS

(5) under the physically significant boundary conditions

$$\left. \frac{\partial^n u}{\partial x^n} \right|_{x \rightarrow \pm \infty} \rightarrow 0, \quad \left. \frac{\partial^n v}{\partial x^n} \right|_{x \rightarrow \pm \infty} \rightarrow 0, \quad n = 0, 1, 2, \dots \quad (6)$$

Because the integrability and the soliton solutions of the model (4) have been known in literature, we always assume $c_1 \neq 0$ except for the special cases which will be particularly pointed out if it is necessary.

In the section 2 of this paper, the non-Painlevé integrability of the GBQS is studied by using the standard Weiss-Tabor-Carnevale (WTC) approach[7]. A special explicit solitary wave solution with a specific velocity selection is given by the truncated Painlevé expansion. In Sec. 3, after reflecting the problem to find the possible solitary waves to the possible motions of the Newtonian type quasi-particles moving in some proper potential fields, the velocity prohibition and selection phenomena for the solitary waves of the GBQS are discussed. For the general solitary wave with allowed velocities, an implicit form of the exact solitary wave solutions is given. Some special solitary wave solutions are plotted also according to the implicit expression. The last section is a short summary and discussions.

2. NON-PAINLEVÉ INTEGRABILITY OF THE GBQS AND ITS EXPLICIT EXACT SOLITARY WAVES WITH THE FIRST TYPE OF SPECIAL VELOCITY SELECTION

2.1 Non-Painlevé integrability of the GBQS with $c_1 \neq 0$

In the modern soliton theory, the study of the Painlevé property[7] plays a very important role because it can be used not only to isolated out (Painlevé) integrable models[8] but also to find many other integrable properties such as the Bäcklund transformations, Lax pair, Schwarzian form etc[7, 5]. Furthermore, even if the model is non-Painlevé integrable, the method can still be used to find some useful things such as the special exact explicit solutions[9]. Because it has been known that the model (4) (i.e., (5) with $c_1 = 0$) is integrable and the general real physical model may require $c_1 \neq 0$, say, (2) and (3), we only check the Painlevé property of the model with $c \neq 0$ in this subsection and find a special solitary wave solution from the next subsection.

The Painlevé property of a partial differential equation system is defined as all the solutions of the system are free of the essential and branch movable singularities around an arbitrary (both characteristic and non-characteristic) singular manifold[10].

According to the above definition of the Painlevé property, the general solutions of the GBQS should have the following expansion around the arbitrary singular manifold

$$\phi \equiv \phi(x, y, t) = 0$$

$$u = \sum_{j=0}^{\infty} u_j \phi^{j+\alpha}, \quad v = \sum_{j=0}^{\infty} v_j \phi^{j+\beta} \quad (7)$$

where $\{\alpha, \beta\}$ should be negative finite integers. Furthermore, if the model possesses the Painlevé property, there should be at least one primary branch (one possible selection of $\{\alpha, \beta, u_0, v_0\}$) such that enough arbitrary functions (six for the GBQS (5), ϕ and five of $\{u_j, v_j\}$) can be included in the Painlevé expansions.

By using the standard leading order analysis, i.e. substituting $u = u_0 \phi^\alpha$, $v = v_0 \phi^\beta$ into the GBQS and balancing the leading terms, we can find that there is only one possible selection

$$\alpha = -2, \quad \beta = -2, \quad u_0 = 12c_1 \phi_x \phi_t, \quad v_0 = 6c_2 \phi_x^2 \quad (8)$$

for $c_2 \neq 0$. For $c_2 = 0$, there is no possible negative integer selection for α and β .

Substituting the above expansion(7) with (8) to Eq.(5) and vanishing all the coefficients of the powers ϕ^k , we can find the following recursion relation for determining the expansion functions u_j and v_j

$$(j-4)(j-6)(1+j)u_j = F_{1j}, \quad (9)$$

$$\phi_x(j-4)c_2 j(j-5)u_j - 12\phi_t(j-4)c_1 v_j = F_{2j}, \quad (10)$$

where F_{1j} and F_{2j} are all only functions of $u_0, u_1, \dots, u_{j-1}, v_0, v_1, \dots, v_{j-1}$ and the derivatives of ϕ . From the recursion relations (9) and (10), we know that all the functions u_j and v_j are fixed by the recursion relations except for those related to the resonance points determined by vanishing the coefficient determinant of (9) and (10):

$$\begin{vmatrix} (j-4)(j-6)(1+j) & 0 \\ \phi_x(j-4)c_2 j(j-5) & -12\phi_t(j-4)c_1 \end{vmatrix} = -12\phi_t c_1 (j-4)^2 (j+1)(j-6) = 0. \quad (11)$$

From the expression (11) and by checking the resonance conditions at the resonant points $j = 4, 4$ and 6 , we know that there are two facts which destroy the Painlevé integrability of the GBQS with $c_1 \neq 0$. The first one is that there is no primary branch at all because the both equations of the QBQS (5) are three order, so the primary branch should have six resonant points such that six arbitrary functions can be entered into the general Painlevé expansion solution. The second one is the resonance conditions at $j = 4$ and $j = 6$ are not satisfied. The lack of the primary branch of the model means some kinds of the logarithmic branch and/or essential singularities around the singular manifold $\phi = 0$ may appear for the GBQS (5) with $c_1 \neq 0$.

Actually, substituting $u = u_0\phi^\alpha$, $v = v_0\phi^\beta$ into the GBQS (5), we can find that there exists a possible non-completely negative selection for the constants α and β :

$$\alpha = -2, \quad \beta = 0, \quad (12)$$

for both $c_2 \neq 0$ and $c_2 = 0$ cases! The allowance of the selection $\beta = 0$ means that the logarithmic term $\ln \phi$ should be included in the expansion of the function u to balance the leading order terms of the first equation of (5).

According to the above discussions, we can conclude that the GBQS with $c_1 \neq 0$ has no Painlevé property. In other words, the model is non-Painlevé integrable. Though the model is non-Painlevé integrable, it is still possible to find some useful information from the Painlevé analysis. In the next subsections we are specially interested to find the possible solitary wave solution from the truncated Painlevé expansion.

2.2 Truncated Painlevé expansion and explicit exact solitary waves with special velocity selections

According to the discussion of the last subsection, we know that the standard truncated Painlevé expansion has the form

$$u = u_0\phi^{-2} + u_1\phi^{-1} + u_2, \quad v = v_0\phi^{-2} + v_1\phi^{-1} + v_2, \quad (13)$$

where $\{u_2, v_2\}$ is a seed solution of the GBQS (5) and usually is taken as constant solution to get the single travelling solitary wave solution of the model.

As usual, substituting the truncated expansion (13) with constants u_2 and v_2 into the GBQS (5) and vanishing the coefficients for different orders of ϕ , we have

$$u_0 = 12c_1\phi_x\phi_t, \quad v_0 = 6c_2\phi_x^2, \quad (14)$$

$$u_1 = -\frac{12}{5} \left(\frac{\phi_x\phi_{tt}}{\phi_t} + \frac{\phi_t\phi_{xx}}{\phi_x} + 3\phi_{xt} \right), \quad (15)$$

$$v_1 = \frac{2}{5} \left(2\frac{\phi_x^2\phi_{tt}}{\phi_t^2} - \frac{4\phi_x\phi_{xt}}{\phi_t} - 13\phi_{xx} \right), \quad (16)$$

for the functions u_0, v_0, u_1, v_1 while the function ϕ should be determined from the following over-determined system

$$u_2u_{1x} + v_{1x} + u_{1t} - c_1u_{1xxt} = 0, \quad (17)$$

$$v_2u_{1x} + v_{1t} + u_2v_x + u_x - c_2u_{1xxx} = 0, \quad (18)$$

$$\begin{aligned} & u_{0t} - u_1\phi_t + c_1(2\phi_xu_{1xt} + u_{1t}\phi_{xx} + (u_1\phi_t)_{xx} - u_{0xxt}) \\ & + u_1u_{1x} - u_2u_1\phi_x + u_2u_{0x} - v_1\phi_x + v_{0x} = 0, \end{aligned} \quad (19)$$

$$(u_0 u_1)_x + 2c_1[\phi_{xx} u_{0t} + (\phi_t u_0)_{xx} - \phi_{xx} \phi_t u_1 - u_{1t} \phi_x^2] - 2u_0 \phi_t + [24c_1(u_{0t} - u_1 \phi_t)_x - 2v_0 - u_1^2 - 2u_2 u_0] \phi_x = 0, \quad (20)$$

$$c_2[3(\phi_x u_{1x})_x + u_1 \phi_{xxx} - u_{0xxx}] - (u_1 + u_1 v_2 + v_1 u_2) \phi_x + (1 + v_2) u_{0x} + v_{0t} + v_{0x} u_2 + (v_1 u_1)_x - v_1 \phi_t = 0, \quad (21)$$

and

$$(u_0 v_1 + v_0 u_1)_x - 2v_0 \phi_t - 2(v_2 u_0 + v_0 u_2 + v_1 u_1 + u_0) \phi_x + 2[u_0 \phi_{xxx} - 3\phi_x (u_1 \phi_x)_x + 3(\phi_x u_0)_x] c_2 = 0. \quad (22)$$

To find all the possible exact solutions of the over-determined system is quite difficult. However, it is a quite easy work to find the travelling solitary wave from the system (14)–(22). For the travelling wave solution $\phi = \phi(x - ct) \equiv \phi(\tau)$, we have

$$\phi_x = \phi_\tau, \quad \phi_t = -c\phi_\tau, \quad (23)$$

with c being an arbitrary velocity parameter.

Under the travelling wave solution condition (23), (14)–(16) are simplified to

$$u_0 = -12c_1 c \phi_\tau^2, \quad v_0 = 6c_2 \phi_\tau^2, \quad u_1 = 12cc_1 \phi_{\tau\tau}, \quad v_1 = -6c_2 \phi_{\tau\tau}. \quad (24)$$

Substituting (24) into (17) and (18) yields

$$2c^2 c_1^2 \phi_{\tau\tau\tau\tau} + (2u_2 cc_1 - 2c^2 c_1 - c_2) \phi_{\tau\tau} = 0, \quad (25)$$

$$2cc_1 c_2 \phi_{\tau\tau\tau\tau} + [2cc_1(1 + v_2) + c_2(c - u_2)] \phi_{\tau\tau} = 0. \quad (26)$$

The equation system (25) and (26) is consistent only for

$$v_2 = \frac{c_2^2 + 2c^2 c_1^2 (c_2 k^2 - 2)}{4c^2 c_1^2}, \quad u_2 = \frac{c_2^2 + 2c^2 c_1^2 (c_2 k^2 - 2)}{2cc_1 c - 2 + 2c_1 c^2 (1 - c_1 k^2)} \quad (27)$$

where the constant k is introduced for convenience later. Using the constant relation (27), the general solution of (25) (and (26)) reads

$$\phi = b_0 + b_1 \tau + b_2 \tau^2 + a_1 \exp(k\tau) + a_2 \exp(-k\tau) \quad (28)$$

with b_0 , b_1 , b_2 , a_0 and a_1 being arbitrary constants. Substituting (24) with (28) into (19), we find that the constants appeared in (28) have to be fixed by

$$b_1 = b_2 = a_1 a_2 = 0. \quad (29)$$

Furthermore, without loss of generality, the constant a_2 can be taken as zero because k is arbitrary (can be taken as both positive and negative). After finishing some simple

direct calculation, one can find that (28) with (29) solves all other remained equations (20)–(22).

Substituting (24), (27), with (28), (29) and $a_2 = 0$ into (13), we get a special solitary wave

$$u = \frac{c_2^2 + 2c^2c_1^2(c_2k^2 - 2)}{2cc_1c - 2 + 2c_1c^2(1 - c_1k^2)} + 3cc_1k^2\text{sech}^2\left[\frac{k}{2}(\tau - \tau_0)\right], \quad (30)$$

$$v = \frac{c_2^2 + 2c^2c_1^2(c_2k^2 - 2)}{4c^2c_1^2} - \frac{3}{2}c_2k^2\text{sech}^2\left[\frac{k}{2}(\tau - \tau_0)\right]. \quad (31)$$

where $\tau_0 = \ln(b_0) - \ln(a_1)$. In general, the boundary conditions expressed in (6) can not be satisfied for the solitary wave solution expressed by (30) and (31). In order to satisfy the boundary conditions (6) for (30) and (31), we have to fix the constants k and c as

$$k^2 = \frac{c_0^2 + 2}{2c_2}, \quad c^2 = \frac{c_0^4}{(2 - c_0^2)}, \quad c_0^2 \equiv \frac{c_2}{c_1}, \quad (32)$$

and then the solitary wave solution becomes

$$u = \pm \frac{3(c_0^2 + 2)}{2\sqrt{2 - c_0^2}}\text{sech}^2\left[\frac{\sqrt{2c_1(c_0^2 + 2)}}{4c_1c_0}\left(x \mp \frac{c_0^2}{\sqrt{2 - c_0^2}}t - \tau_0\right)\right], \quad (33)$$

$$v = -\frac{3}{4}(c_0^2 + 2)\text{sech}^2\left[\frac{\sqrt{2c_1(c_0^2 + 2)}}{4c_1c_0}\left(x \mp \frac{c_0^2}{\sqrt{2 - c_0^2}}t - \tau_0\right)\right]. \quad (34)$$

From the above discussions, the zero boundary travelling solitary wave solution (33) (and (34)) obtained via the truncated Painlevé expansion is valid only for the GBQS (5) with the condition $c_0^2 < 2$ (i.e., $c_2 < 2c_1$) and $c_2 \neq 0$ for $c_1 \neq 0$. The solitary wave solution (33) (and (34)) possesses only two special isolated velocities $\pm \frac{c_0^2}{\sqrt{2 - c_0^2}}$. Is there any other travelling solitary wave with other velocities for the GBQS (5)? The answer is positive because in the truncated Painlevé expansion approach we require all the coefficients of the different powers of ϕ being zero. This strong condition leads to the loss of the generality. In the next section, we discuss this problem generally by casting the problem to the possible motions of a Newtonian classical quasi-particle in some possible potentials.

3. IMPLICIT TRAVELLING SOLITARY WAVE SOLUTIONS AND SPECIAL VELOCITY SELECTIONS AND PROHIBITIONS

From the last section, we know that by means of the truncated Painlevé expansion approach, we can only obtain a special exact solitary wave solution with the boundary

condition (6). Actually, the travelling solitary waves with different velocities do exist for the GBQS (5). Though we can not explicitly write down the exact solitary wave solutions with the boundary condition (6) for the different velocities, we can find an uniform and implicit formula for the solitary wave solutions except for some isolated special critical velocities.

For a travelling wave solution, $u = u(x - ct) \equiv u(\tau)$, $v = v(\tau)$, the motion equation system (5) becomes an ordinary differential equation system

$$\begin{aligned} -cu + \frac{1}{2}u^2 + v &= -c_1cu_{\tau\tau} \\ -cv + (1 + v)u &= c_2u_{\tau\tau}, \end{aligned} \quad (35)$$

where the both equations have been integrated once with respect to τ and the integration constants have been fixed as zero because of the boundary conditions given in (6). From the first equation of (35), we know that the travelling wave of the field v can be simply expressed by

$$v = -\frac{1}{2}(-2cu + u^2 + 2c_1cu_{\tau\tau}). \quad (36)$$

Substituting it into the second equation of (35) we have

$$u_{\tau\tau} = \frac{2(1 - c^2)u - u^3 + 3cu^2}{2c_1(c_0^2 - c^2 + cu)}. \quad (37)$$

The first integration of (37) reads

$$\begin{aligned} u_\tau^2 &= \frac{1}{c_1} \left(\frac{c_0^2}{c^2} - 1 \right) \left(\frac{c_0^4}{c^2} + c_0^2 - 2 \right) \ln \frac{cu + c_0^2 - c^2}{c_0^2 - c^2} \\ &\quad - \frac{u^3}{3c_1c} + \frac{u^2}{2c_1} \left(2 + \frac{c_0^2}{c^2} \right) - \frac{u}{c_1c} \left(\frac{c_0^4}{c^2} + c_0^2 - 2 \right) \\ &\equiv -2V(u) \end{aligned} \quad (38)$$

for $c \neq 0$ and

$$u_\tau^2 = -\frac{u^4}{4c_2} + \frac{u^2}{c_2} \equiv -2V0(u), \quad \tau = x, \quad (c_2 \neq 0) \quad (39)$$

for $c = 0$. To get the relations (38) and (39), the integration constants have been fixed appropriately such that the boundary conditions (6) may be satisfied.

Now, from the expressions (37)–(39), it is known that to find the possible travelling solitary wave solutions of the GBQS (5) with boundary condition (6) is equivalent to find the possible special motions of a classical quasi-particle moving in the potential fields $V(u)$ and/or $V0(u)$ related to the *maximum* point at $u = 0$ with the “space” variable u and the “time” variable τ [11].

If the existence problem of the solitary wave is solved, except for some special critical cases (see later), the travelling solitary waves with the boundary condition (6) can be uniformly expressed implicitly by

$$\tau - \tau_0 = \pm \int^u \frac{du}{\sqrt{-2V(u)}}, \quad (40)$$

where $V(u)$ is defined in (38) and the integration constant τ_0 is related to the location of the solitary wave.

To see the possible solitary wave solutions qualitatively, the possible motions of the quasi-particle in the potentials $V(u)$ and/or $V_0(u)$, it is convenient to study the structures of the potentials $V(u)$ and $V_0(u)$ at the same time.

To study the structures of the potentials $V(u)$, we firstly isolate out some special and/or critical cases.

The first special case is same as that in (32). When the velocity c of the solitary wave is fixed by (32), the logarithmic term of (38) vanishes and the function u can be explicitly integrated out. The result is reasonably same as that obtained in the last section via the truncated Painlevé analysis. The potential structure at this special case is plotted in Fig. 1a for the special model (3), i. e. $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{6}$ and then $c_0 = \sqrt{\frac{1}{3}}$ and $c = \sqrt{\frac{1}{15}}$. The corresponding soliton solution for the field u is plotted in Fig.1b.

The second special critical case is that if the velocity is determined by

$$c = \pm c_0, \quad (41)$$

then the logarithmic term of (38) is also vanished. However, in this case there is no solitary wave solution with the boundary condition (6) except for $c_0 = 1$, i.e., $c_1 = c_2$. In other words, the solitary wave solutions with the velocities $\pm c_0$ are prohibited for the GBQS (5) with $c_2 \neq c_1$, $c_1 \neq 0$. This velocity prohibition property can be observed more clearly from (37). After substituting (41) into (37), one can directly see that it is impossible to satisfy the boundary condition (6) except for $c_0 = 1$. While if $c_0 = 1$, (41) is same as (32) and then the related solitary wave solution is also given by (33). The related potential structure of the potential $V(u)$ with $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{6}$ and then $c = c_0 = \sqrt{\frac{1}{3}}$ is plotted in Fig. 2. From Fig. 2, we can see also that when $c = c_0$, $u = 0$ is not an extremum of the potential while a solitary wave is corresponding to a possible special motion of the quasi-particle related to a maximum of the potential.

The third special critical case is related to the static solitary wave solution. For the static solitary wave solution of the GBQS, two subcases, $c_2 > 0$ and $c_2 \leq 0$, should be

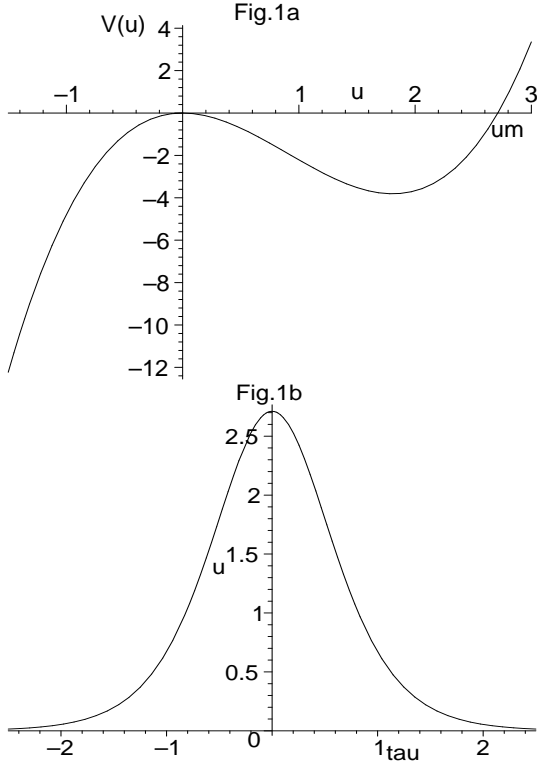


FIGURE 1. (a). The potential structures for the special velocity selection $c = \frac{c_0^2}{\sqrt{2-c_0^2}}$ with $c_1 = \frac{1}{2}$ and $c_0 = \frac{\sqrt{3}}{3}$ ($c_2 = \frac{1}{6}$). (b) The bell shape solitary wave solution related to (a). All the figures of this paper have no unit because the model is dimensionless. $\tau \equiv \tau$ in all figures also.

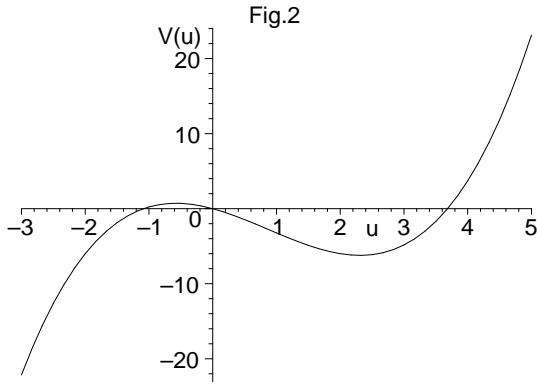


FIGURE 2. The potential structures for the critical velocity $c = c_0$ with $c_1 = \frac{1}{2}$ and $c_0 = \frac{\sqrt{3}}{3}$.

clarified. For the GBQS (5) with $c = 0$, $c_2 \neq 0$, (38) is not valid and has to be changed as (39). The corresponding structure for the potential $V_0(u)$ is plotted in Fig. 3a for $c_2 = \frac{1}{6} > 0$ and the corresponding solitary wave solution is plotted in Fig. 3b.

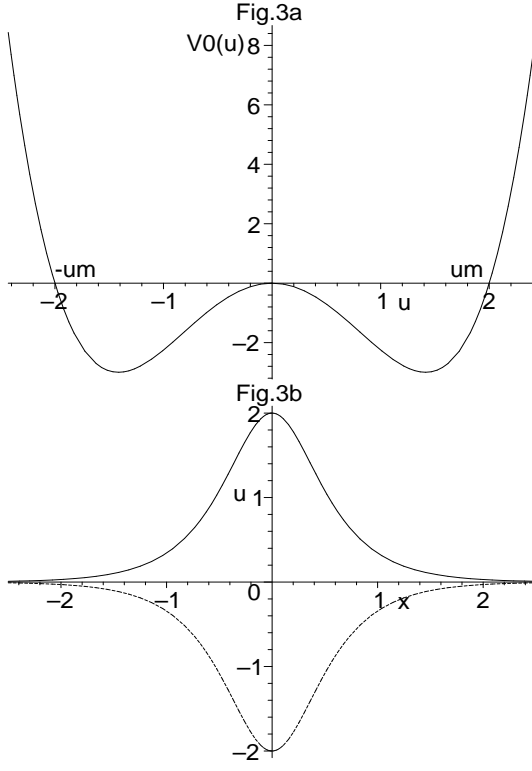


FIGURE 3. (a) The potential structures for the static velocity $c = 0$ with $c_2 = \frac{1}{6} > 0$; (b) The related static bell shape solitary wave (up solid line) and ring shape wave (the lower dashed line).

From (39) and (36), it is straightforward to get that the static solitary wave of the GBQS with $c_2 > 0$ has the form

$$u = \pm 2 \operatorname{sech} \frac{x}{\sqrt{c_2}}, \quad v = -\operatorname{sech}^2 \frac{x}{\sqrt{c_2}}. \quad (42)$$

Fig. 4 shows the structure for the potential $V0(u)$ with $c_2 = -\frac{1}{3} < 0$ which includes the integrable case (4). From (39) and Fig. 4, it is also easy to see that when $c_2 < 0$, $u = 0$ is a minimum of the potential $V0(u)$. So there is no static solitary wave of the GBQS with $c_2 < 0$ and zero boundary conditions because a solitary wave corresponding to the motion of the classical quasi-particle in the potential field related to a *maximum*. Furthermore, from (35) with $c = 0$ and $c_2 = 0$, we can also know that there is no static solitary wave at all. In summary, the static solitary wave with zero boundary condition is prohibited for the GBQS with $c_2 \leq 0$. Especially, there is no static solitary wave with zero boundary conditions for the systems (2) and (4) even if (4) is integrable.

The fourth type of velocity selection is related to

$$c = \pm 1, \quad c_0 \neq 1. \quad (43)$$

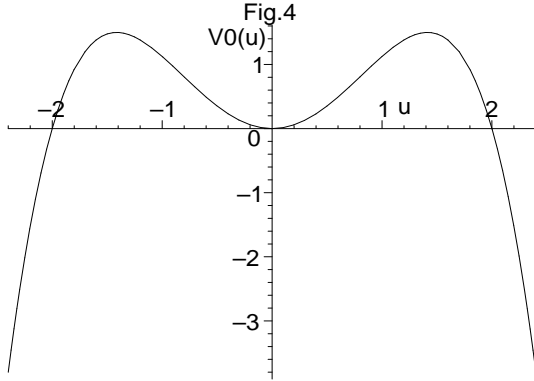


FIGURE 4. The potential structures for the static velocity $c = 0$ with $c_2 = -\frac{1}{3} < 0$.

Usually, the solitary waves are decay exponentially apart from the solitary wave center. However, for the GBQS with $c_2 \neq c_1$, if the solitary wave possesses the velocities ± 1 then it decays only algebraically. This velocity selection property can be seen from the second order differentiation with respect to u of the potential $V(u)$ at the point $u = 0$. The second order differentiation of $V(u)$ for $c^2 = 1$ reads

$$\frac{d^2V(u)}{du^2} = -\frac{u[6(c-u-cc_0^2)+2cu^2+3uc_0^2]}{2c_1(1-cu-c_0^2)^2}, \quad (c^2 = 1). \quad (44)$$

It is known that the boundary values ($u = 0$ in our case) of exponentially decayed solitary wave solutions are linked with the maxima of the potential $V(u)$ [11]. However, from (44) we know that when the velocity $c = \pm 1$, $u = 0$ is only an inflexion point of the potential $V(u)$. It is also known that an inflexion point may be related to a rational (or algebraic) solitary wave solution[12]. This type of the algebraic solitary wave is similar to some what of the solitary wave at the critical point where the phase transition occur for some types of quantum fields and condense matter systems[12].

To see the algebraic decay property of the solitary waves at this critical case, we take $c = 1$, $c_1 = \frac{1}{2}$ and $c_0 = 50$. Under this parameter selection, the related potential becomes

$$V(u) = 15624992502 \ln\left(\frac{u}{2499} + 1\right) - \frac{1}{3}u^3 + 1251u^2 - 6252498u \equiv V1(u) \quad (45)$$

$$\approx \frac{1}{2499}u^3 - \frac{139}{1387778}u^4 \equiv V2(u). \quad (46)$$

The corresponding structure for the exact potential $V1(u)$ and the approximate potential $V2(u)$ are plotted at the same figure, Fig. 5a. From Fig. 5a, it is seen that two lines are almost coincide with each other at the plotted region which related to the

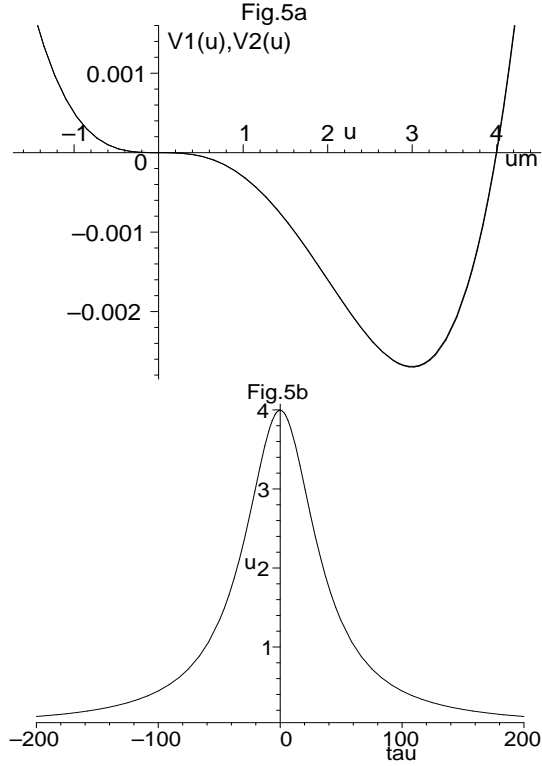


FIGURE 5. (a). The potential structures for the potentials $V1(u)$ and $V2(u)$ at the critical velocity $c = 1$ with $c_1 = \frac{1}{2}$ and $c_0 = 50$. (b). The corresponding approximately algebraic decayed solitary wave.

quasi-particle move in. Actually, the approximation (46) is quite well up to $u \sim \pm 1000$ while the solitary wave is located only at the region $u = 0 \sim 4$. So to see the solitary wave structure in this special case we can safely use the approximate potential $V2(u)$ to calculate the related solitary wave. The result reads

$$u_{c=1, c_1=0.5, c_2=100} \approx \frac{4998}{\tau^2 + 1251}. \quad (47)$$

Fig. 5b. shows the structure of the algebraic solitary wave expressed by (47).

From the analysis of the existence problem of the solitary waves at the critical velocities, we know that for different velocities the solitary waves may possess different shapes. This type phenomena occurs also in integrable cases. For instance, for the bidirectional Kaup-Kupershmidt equation, its right moving solitons possess single-humped shape while its left moving solitons possess double-humped shape[13]. In integrable case, the velocity prohibition phenomena are also common. For instance, for the system (4) the static solitary wave with zero boundary condition is prohibited and for the

KdV equation (1), all the left moving and static soliton solutions with zero boundary conditions are prohibited.

Now the important problem is in addition to the selections and prohibitions at the critical points whether there exist further prohibitions for some other velocity regions. To solve this problem, we restrict ourselves for $c_1 > 0$ (because most of the real physical systems, say, (2) and (3), possess this property) and $c > 0$ (because the symmetry property $\{c, u\} \leftrightarrow \{-c, -u\}$ of (37)).

To find the existence conditions of the solitary waves of the GBQS (5) with zero boundary conditions (6) for noncritical cases, we can use the maximum condition, $\frac{d^2V(u)}{du^2}\Big|_{u=0} < 0$ and the real condition of the potential $V(u)$.

The maximum condition of the potential $V(u)$ at $u = 0$ reads

$$\frac{d^2V(u)}{du^2}\Big|_{u=0} = \frac{1 - c^2}{c_1(c^2 - c_0^2)} < 0, \quad (48)$$

and the real condition of $V(u)$, reads

$$\frac{cu}{c_0^2 - c^2} + 1 > 0. \quad (49)$$

After finishing the detail analysis with help of the conditions (48) and (49), we find the following six different structures for the potential $V(u)$ for $c_1 > 0$:

Case 1.

$$0 < c^2 < c_0^2 < 1. \quad (50)$$

When c^2 , the square of the velocity parameter, is located in the range $(0, c_0)$ with $c_0 < 1$, the related potential $V(u)$ possesses the structure as shown in Fig. 6a for $c > 0$. From Fig. 6a we know that there exist a special motion for a classical quasi-particle moving in this potential related to the maximum at $u = 0$: At the beginning ($\tau = -\infty$), the quasi-particle is located at the peak center $u = 0$, as time “ τ ” increases, the quasi-particle “roll” down the hill up to the $um \equiv u_{am}$ point and then return back to the original point $u = 0$ at “time” $\tau = +\infty$. This type of special motion of the quasi-particle is just related to the solitary wave solution of the GBQS (5) with (50). From Fig. 6a we know also that the quasi-particle can roll only to right (the positive u direction) that means the right moving solitary wave ($c > 0$) possesses bell shape for the field u . By using the invariant transformation of (37), $\{u, c\} \rightarrow \{-u, -c\}$, we know that the left moving solitary wave ($c < 0$) possesses ring shape. The corresponding exact solitary wave with the same parameters as shown in Fig. 6a is plotted in Fig. 6b.

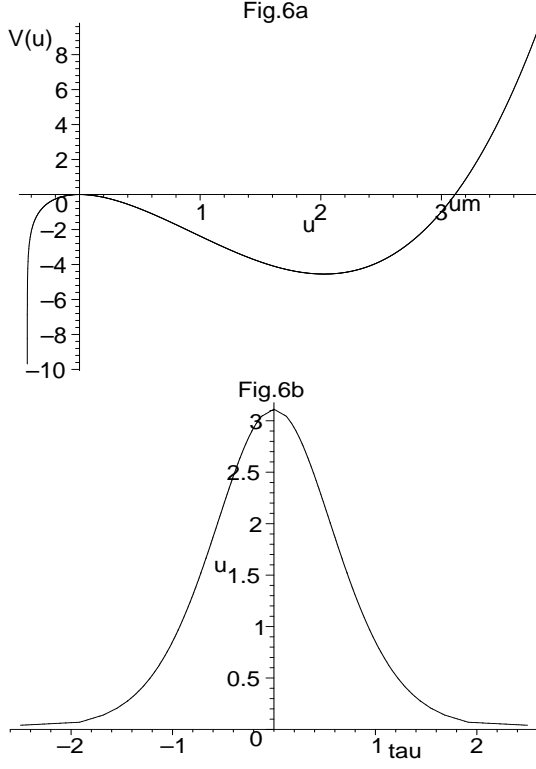


FIGURE 6. (a) The typical structure for the potential of the quasi-particle located in with $c_1 > 0$ and $c \geq 0$, $0 < c^2 < c_0^2 < 1$ and $\{c_0 = \frac{1}{\sqrt{3}}, c = 0.4, c_1 = \frac{1}{2}\}$. (b) The corresponding solitary wave solution related to (a).

Case 2.

$$c_0^2 < c^2 < 1. \quad (51)$$

In this case, the related potential $V(u)$ possesses the structure as shown in Fig. 7 for $c > 0$. From Fig. 7 we know that there is no solitary wave solution with zero boundary condition in this case because $u = 0$ is only a minimum of the potential. If the quasi-particle stay at $u = 0$ at the beginning then it can only be stayed there forever. In other words, the solitary waves of the GBQS (5) with the velocities $c_0^2 < c^2 < 1$ and zero boundary condition (6) are totally prohibited.

Case 3.

$$c_0^2 < 1, c^2 > 1. \quad (52)$$

The structure of the potential $V(u)$ for $c^2 > 1$ with $c_0^2 < 1$ possesses the form as shown in Fig. 8a. Similar to the case 1, there exist the right moving bell shape solitary waves and the left moving ring shape solitary waves for the field u . The right moving bell shape solitary wave solution related to Fig. 8a is plotted in Fig. 8b.

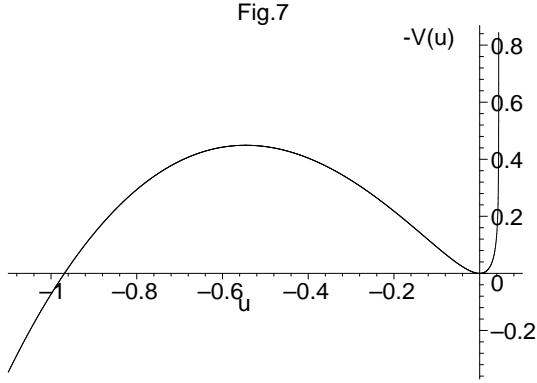


FIGURE 7. The typical structures for the potential with $c_1 > 0$ and $c \geq 0$ $c_0^2 < c^2 < 1$ and $\{c_0 = \frac{1}{\sqrt{3}}, c = 0.6, c_1 = \frac{1}{2}\}$.

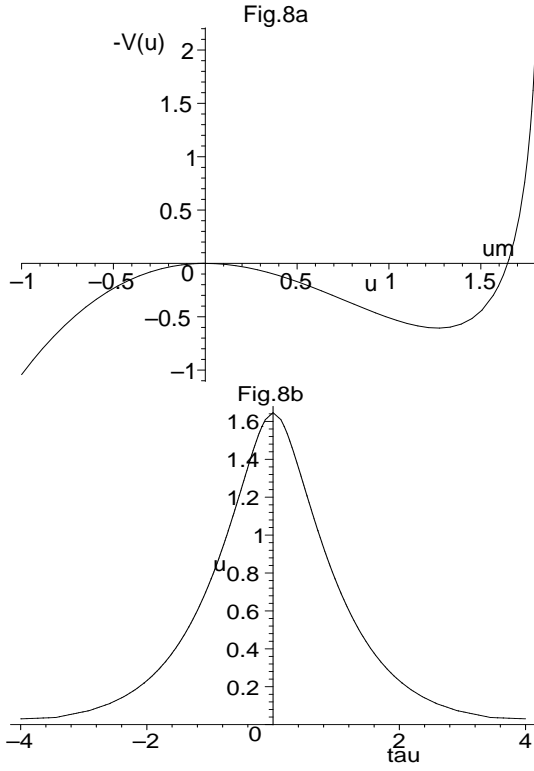


FIGURE 8. (a). The typical structures for potential $V(u)$ with $c_1 > 0$, $c \geq 0$, $c_0^2 < 1$, $c^2 > 1$ and $\{c_0 = \frac{1}{\sqrt{3}}, c = 2, c_1 = \frac{1}{2}\}$. (b). The corresponding solitary wave related to (a).

In the first three cases, c_0 is less than 1 and the special models (2) and (3) satisfy this condition. When this condition is not satisfied, we have three other different potential structures.

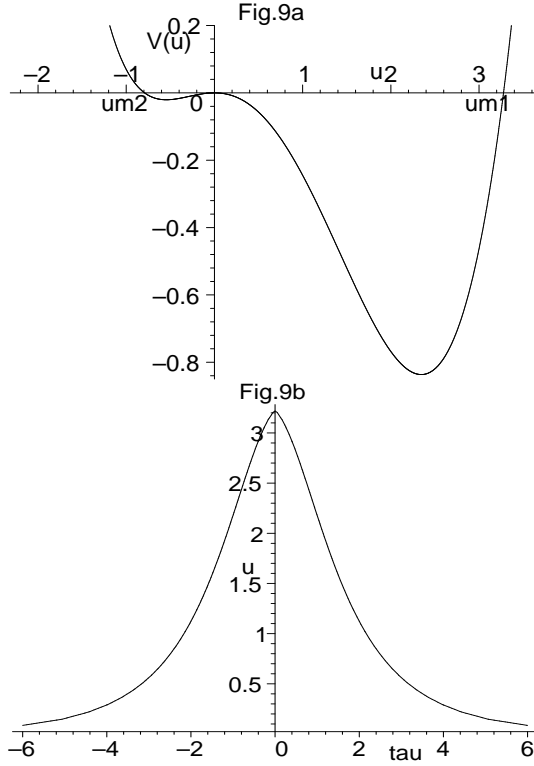


FIGURE 9. (a). The typical structure for potential of the quasi-particle located in with $c_1 > 0$, $c \geq 0$, $c_0^2 > 1$, $c^2 < 1$ and $\{c_0 = 2, c = 0.6, c_1 = \frac{1}{2}\}$. (b). The corresponding solitary wave related to (a).

Case 4.

$$0 < c^2 < 1 < c_0^2. \quad (53)$$

The corresponding potential structure of the fourth case (53) is plotted in Fig. 9a. Different from the first and third cases, from Fig. 9a we know the quasi-particle located the $u = 0$ hill may roll down the hill to both the left and right sides and finally return back to the hill. So in this case, the right moving solitary waves may have both bell and ring shapes and the left moving solitary waves possess the same property. The right moving bell shape solitary wave solution related to this case with the same parameters as Fig. 9a is plotted in Fig. 9b. From Fig. 9a, we can know also that for the right moving solitary waves, the bell shape solitary waves possess larger amplitudes and for the left moving solitary waves, the ring shape solitary waves possess larger amplitudes.

Case 5.

$$1 < c^2 < c_0^2. \quad (54)$$

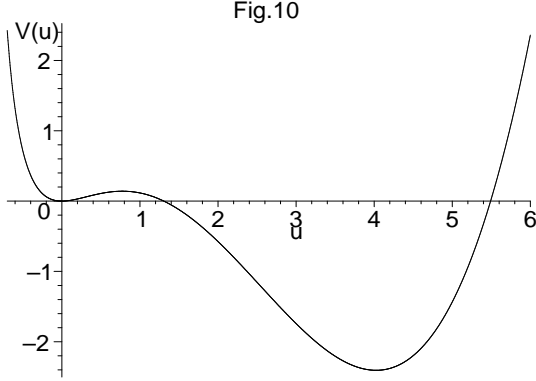


FIGURE 10. The typical structures for potential $V(u)$ with $c_1 > 0$, $c \geq 0$, $1 < c^2 < c_0^2$ and $\{c_0 = 2, c = 1.6, c_1 = \frac{1}{2}\}$.

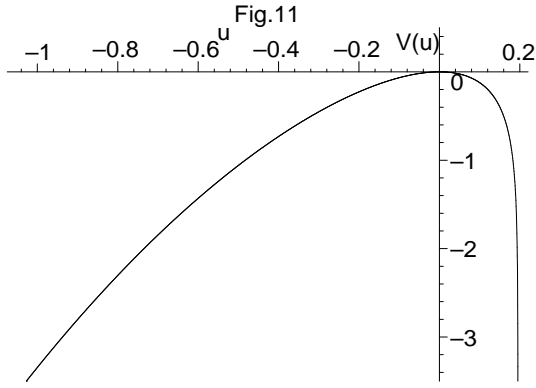


FIGURE 11. The typical structure for potential $V(u)$ with $c_1 > 0$, $c \geq 0$, $1 < c_0^2 < c^2$ and $\{c_0 = 2, c = 2.1, c_1 = \frac{1}{2}\}$.

When c^2 is located in the range $(1, c_0)$, the corresponding potential structure possesses the form as shown in Fig. 10. From Fig. 10 we know that $u = 0$ is only a minimum of the potential. So similar to the discussion of the case 2, we can conclude that there is no zero boundary solitary wave for the GBQS with the condition (54).

Case 6.

$$1 < c_0^2 < c^2. \quad (55)$$

For the last case (55), the related potential structure is shown in Fig. 11. From Fig. 11 we know that $u = 0$ is really a maximum of the potential, the quasi-particle may roll down the $u = 0$ hill in both sides, however, it can not be back! That means the fast moving ($c^2 > c_0$) solitary waves for the GBQS with the condition $c_0^2 > 1$ are completely prohibited also.

Combining the critical cases, we summarize the conclusions for the possible solitary waves of the GBQS (5) with zero boundary conditions (6) and $c_1 > 0$ in the following table.

Table 1. Existent and prohibited regions of the solitary waves.

model	velocity	solitary wave	shape
$c_2 < c_1$	$c < -1$	yes	ring shape ($u < 0$)
$c_2 < c_1$	$-1 \leq c \leq - c_0 $	no	
$0 < c_2 < c_1$	$- c_0 < c < 0$	yes	ring shape
$c_2 < 0$	$- c_0 < c < 0$	no	
$0 < c_2 < c_1$	$c = 0$	yes	both ring shape and bell shape
$c_2 < 0$	$c = 0$	no	
$0 < c_2 < c_1$	$0 < c < c_0 $	yes	bell shape ($u > 0$)
$c_2 < 0$	$0 < c < c_0 $	no	
$c_2 < c_1$	$ c_0 \leq c \leq 1$	no	
$c_2 < c_1$	$c > 1$	yes	bell shape
$c_1 < c_2$	$c < -1$	no	
$c_1 < c_2$	$c = -1$	yes	algebraic, ring shape
$c_1 < c_2$	$-1 < c < 1$	yes	both ring shape and bell shape
$c_1 < c_2$	$c = 1$	yes	algebraic, bell shape
$c_1 < c_2$	$c > -1$	no	

4. SUMMARY AND DISCUSSIONS

In summary, using the standard WTC's Painlevé PDE test, we prove that the GBQS is non-Painlevé integrable except for the special case for $c_1 = 0$, i.e., (4). For the non-Painlevé integrable GBQS with zero boundary conditions (6), the truncated Painlevé expansion approach leads to only a special $sech^2$ shape solitary wave solution with a special velocity selection. To find all the possible travelling solitary wave solutions of the GBQS with $c_1 > 0$ for all the possible model parameter regions, we map the problem to find the possible motions of a Newtonian classical quasi-particle moving in some possible potentials. After considering all the possible motions of the classical quasi-particle in the possible potentials related to the maxima of the potential at $u = 0$, all the possible travelling solitary wave solutions related to zero boundary conditions are found for all the possible velocities and the model parameter ranges with $c_1 > 0$.

Similar to the integrable cases such as the KdV equation, the solitary waves at some special ranges, the travelling solitary waves with zero boundary conditions are completely prohibited. For the KdV equation, all the left moving and static solitons with zero boundary conditions are prohibited. For the GBQS with $c_1 > 0$, there are three different cases. For the first case, $0 < c_2 < c_1$, both the faster and slower moving solitary waves are allowed while the solitary waves moving in the “middle” velocities $\frac{c_2}{c_1} \equiv c_0^2 \leq c^2 \leq 1$ are completely prohibited. For the second case, $0 < -c_2 < c_1$, only

the faster moving solitary waves are allowed while all the slower moving solitary waves $c^2 < 1$ are prohibited. For the third case $0 < c_1 < c_2$, contrary to the second case, all the slower moving solitary waves are allowed while all the faster moving $c^2 > 1$ solitary waves are completely prohibited. For the first two cases the solitary waves at the critical velocities $c = \pm 1$ are prohibited while for the third case, there are algebraic solitary waves at the critical velocities $c = \pm 1$. For the first two types of GBQS, the right moving solitary waves possess bell shape and the left moving solitary waves possess ring shape. For the third type of GBQS, the right moving bell shape solitary wave possess larger amplitudes than those of the right moving ring shape solitary waves (with the same value of the velocity) while for the left moving solitary waves, the ring shape solitary waves possess larger amplitude.

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